Neap

Trial Examination 2022

HSC Year 12 Mathematics Extension 1

General Instructions	• Reading time – 10 minutes • Working time – 2 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total marks: 70	SECTION I – 10 marks (pages 2–6) • Attempt Questions 1–10 • Allow about 15 minutes for this section SECTION II – 60 marks (pages 7–11) • Attempt Questions 11–14 • Allow about 1 hour and 45 minutes for this section

SECTION I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given $\tan\theta = \dfrac{1}{3}$, what is the exact value of $\tan\left(\theta + \dfrac{\pi}{3}\right)$?

A. $\dfrac{\sqrt{3}+3}{3\sqrt{3}-1}$

B. $\dfrac{\sqrt{3}-3}{3\sqrt{3}+1}$

C. $\dfrac{1+3\sqrt{3}}{3-\sqrt{3}}$

D. $\dfrac{1-3\sqrt{3}}{3+\sqrt{3}}$

2 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to $\displaystyle\int_e^{e^2} \frac{(\ln x)^3}{x}dx$?

A. $\dfrac{1}{2}\displaystyle\int_1^4 u\ du$

B. $2\displaystyle\int_1^4 u\ du$

C. $2\displaystyle\int_1^4 u^{\frac{3}{2}}\ du$

D. $\displaystyle\int_1^4 u^6\ du$

3 Which of the following is an anti-derivative of $\int \frac{4}{\sqrt{9-x^2}}dx$?

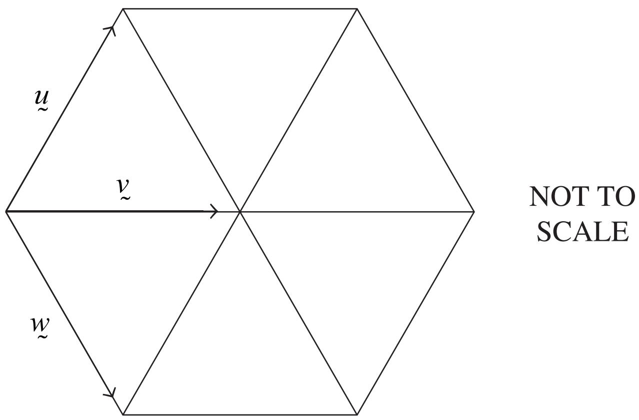
A. $\frac{4}{3}\sin^{-1}\left(\frac{x}{3}\right)+c$

B. $\frac{4}{3}\sin^{-1}(3x)+c$

C. $4\sin^{-1}\left(\frac{x}{3}\right)+c$

D. $4\sin^{-1}(3x)+c$

4 Six equilateral triangles form a hexagon with side lengths of 4 cm. The vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown in the diagram.



Which of the following is the value of $\underset{\sim}{u}\cdot\left(\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}\right)$?

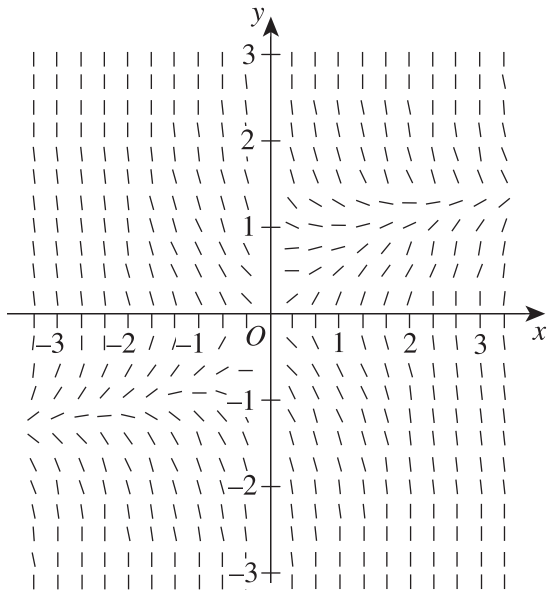
A. 8

B. 16

C. 32

D. 48

5 The slope field for a first order differential equation is shown.



Which of the following best represents the differential equation shown in the slope field?

A. $\dfrac{dy}{dx} = \dfrac{x}{y} - y^2$

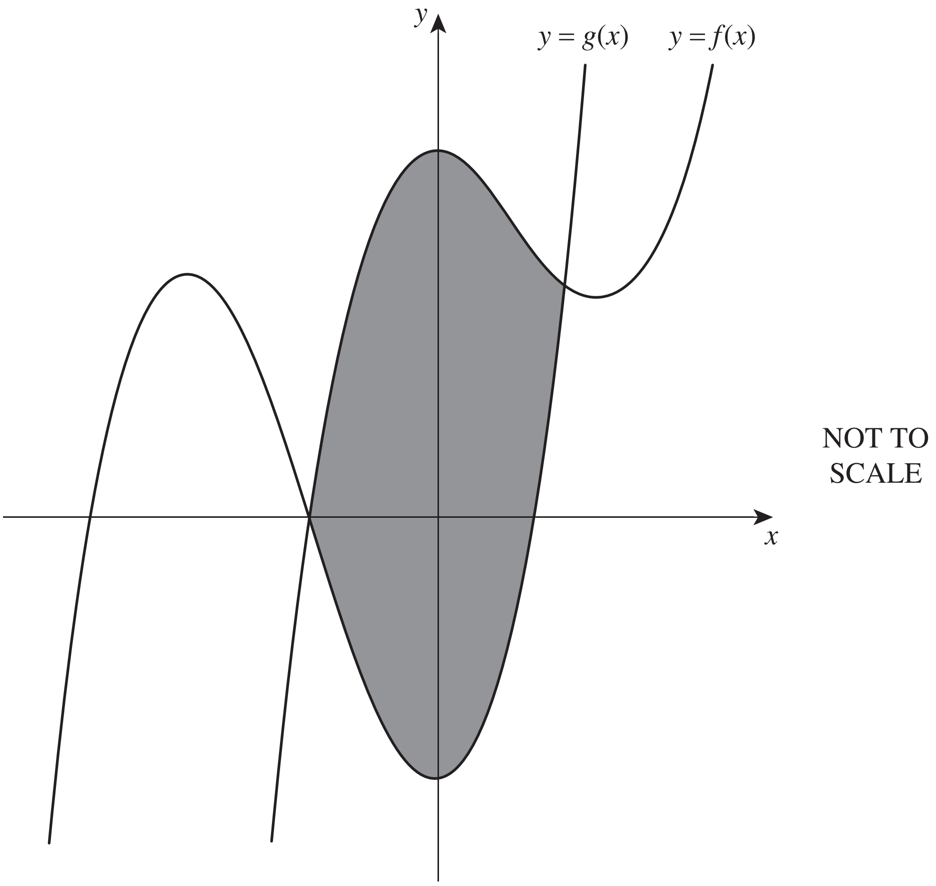
B. $\dfrac{dy}{dx} = \dfrac{x}{y} + y^2$

C. $\dfrac{dy}{dx} = -\dfrac{x}{y} - y^2$

D. $\dfrac{dy}{dx} = -\dfrac{x}{y} + y^2$

6 Consider two curves with the equations $f(x) = x^3 - 2x^2 + 3$ and $g(x) = x^3 + 3x^2 - 2$.

The diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



Which of the following gives the correct expression for the shaded area between the two curves?

A. $\displaystyle\int_{-2}^{3} -5x^2 + 5\, dx$

B. $\displaystyle\int_{-2}^{3} 5x^2 - 5\, dx$

C. $\displaystyle\int_{-1}^{1} 5x^2 - 5\, dx$

D. $\displaystyle\int_{-1}^{1} -5x^2 + 5\, dx$

7 The expression $2\cos x - 3\sin x$ is written in the form $R\cos(x+\theta)$, where $R > 0$ and $0 \le \theta \le \frac{\pi}{2}$.

What is the value of $\tan\theta$?

A. $-\frac{3}{2}$

B. $-\frac{2}{3}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

8 In a Year 12 Mathematics class, the teacher can give one of five grades (A, B, C, D or E) to each student.

What is the minimum number of students required so that six students are guaranteed to receive the same grade?

A. 6

B. 25

C. 26

D. 30

9 Consider the vectors $\underset{\sim}{p} = \begin{pmatrix} t-8 \\ 6 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 3 \\ 2t \end{pmatrix}$.

What are the possible values of t so that $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are parallel?

A. –3, –11

B. –1, 9

C. 1, –9

D. 3, 11

10 A Mathematics department consists of 12 teachers. Nine of the teachers wear glasses and three of the teachers do not wear glasses. Five of these teachers go out to dinner together.

In how many ways will there be more teachers who wear glasses than teachers who do not wear glasses in the group who go out for dinner?

A. 126

B. 220

C. 756

D. 792

SECTION II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$ has a double root at $x = 3$. Find the values of a and b, where a and b are real numbers. **3**

(b) Find the values of n such that $\binom{5n+3}{5n+1} \geq 528$, where n is a positive integer. **3**

(c) A large cylindrical water tank with a base radius of 0.4 m has a tap at the bottom. This tap allows water to flow out of the tank at a rate of $\frac{dV}{dt} = k\sqrt{h}$, where V m^3 is the volume of water, h is the depth of the water in metres, t is the time in minutes and k is a constant.

Initially, the water in the tank is 1 m deep. Twenty minutes after the tap has been turned on, the water in the tank is 0.36 m deep.

(i) Show that $k = -\frac{4}{625}\pi$. **3**

(ii) How long will it take for the tank to empty, correct to the nearest minute? **1**

(d) Consider the expansion of $(2x - p)^9$. The coefficient of the seventh term is –672 000. Find the value of p. **2**

(e) Consider the function $f(x) = x^2 - c^2$, where c is a positive real number. **3**

Sketch the graph of $y = \frac{1}{|f(x)|}$, showing all important features including the turning point(s), intercept(s) and asymptote(s).

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$1 + 2 \times 2^{-1} + 3 \times 2^{-2} + 4 \times 2^{-3} + \ldots + n \times 2^{-(n-1)} = \frac{2^{n+1} - n - 2}{2^{n-1}}.$$

(b) (i) Show that $\dfrac{d}{dx}\left(-\cot x\right) = \dfrac{1}{\sin^2 x}$. **2**

(ii) Use the substitution $x = 4\sin\theta$ and the results from part (i) to show that **4**

$$\int_2^{2\sqrt{3}} \frac{1}{x^2\sqrt{16 - x^2}}\,dx = \frac{\sqrt{3}}{24}.$$

(c) Consider the graph of $f(x) = 2x\sin^{-1} x$, where $-1 \leq x \leq 1$.

(i) Show that $f(x)$ is an even function. **1**

(ii) Hence, sketch the graph of $f(x)$, showing all important features including the intercept(s) and endpoint(s). **2**

(d) A particular instant noodle company states that, at most, 3% of all their noodle packets marked 85 grams may weigh less than 82 grams.

(i) Find the mean and standard deviation for this distribution of sample proportions. Give the mean correct to two decimal places and the standard deviation correct to four decimal places. **1**

(ii) Part of a table of $P(Z < z)$ values, where Z is a standard normal variable, is shown. **2**

z	*0.00*	*0.01*	*0.02*	*0.03*	*0.04*	*0.05*	*0.06*	*0.07*	*0.08*	*0.09*
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8461	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

For a random sample of 250 noodle packets, estimate the probability that 2% of the noodle packets weigh less than 82 grams.

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Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle, P, is projected from point A, which is 147 m above the ground. The particle P has an initial speed of u m/s and is projected at an angle of θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$.

After T seconds, the particle P lands on the ground at point B with a speed of $4u$ m/s.

Acceleration due to gravity is 10 m/s^2.

(i) Find the equations of motion. **3**

(ii) Find the value of u. **3**

(iii) Hence, show that $\sin\theta = \dfrac{5T^2 - 147}{14T}$. **2**

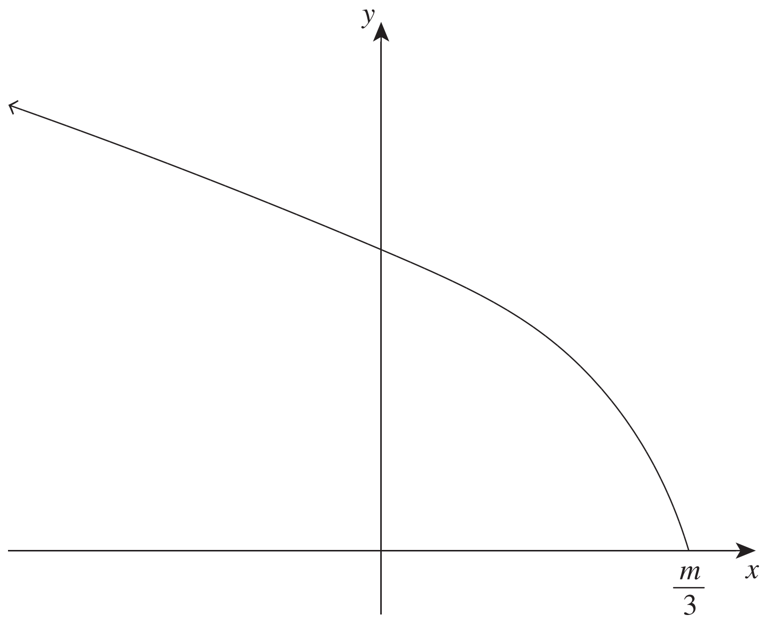
(iv) Hence, or otherwise, show that $\dfrac{\sqrt{735}}{5} < T < 7$. **3**

(b) (i) Use the product rule to show that $y = x\int f(x)\,dx$ is a solution for the differential equation $x\dfrac{dy}{dx} - y = x^2 f(x)$. **2**

(ii) Solve the differential equation $x\dfrac{dy}{dx} - y = x^5$, when $x = 2$ and $y = 5$. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = \sqrt{m - 3x}$ for $x \le \frac{m}{3}$. The graph of $f(x)$ is shown. **4**



NOT TO SCALE

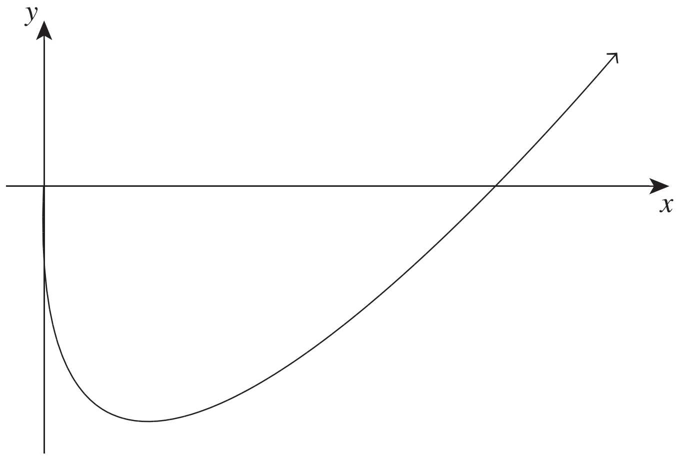
The area enclosed by the graph $f(x)$, the x-axis and the y-axis is rotated about the y-axis.

Find the value of m such that the volume of the solid formed is $\frac{5000\pi}{27}$ units^3.

Question 14 continues on page 11

Question 14 (continued)

(b) Consider the graph of $f(x) = x - 4\sqrt{x} - 1$ for $x \geq k$, such that the inverse function $f^{-1}(x)$ exists.



NOT TO SCALE

(i) Find the value of k. **2**

(ii) Find $f^{-1}(x)$. **2**

(iii) State the domain and range of $f^{-1}(x)$. **2**

(c) (i) Show that $(x + \tan\theta)$ and $(x - \cot\theta)$ are factors of $x^2 - 2(\cot 2\theta)x - 1 = 0$. **2**

(ii) Hence, show that $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{2} - 1$. **2**

(iii) Hence, or otherwise, find the exact value of $\cot\left(\dfrac{\pi}{16}\right) - \tan\left(\dfrac{\pi}{16}\right)$. **1**

End of paper

MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1
MATHEMATICS EXTENSION 2
REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

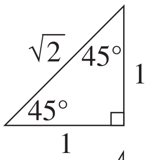
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

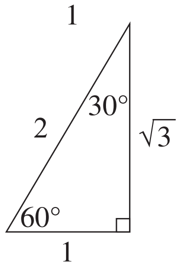
$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$





Trigonometric identities

$\sec A = \frac{1}{\cos A}, \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

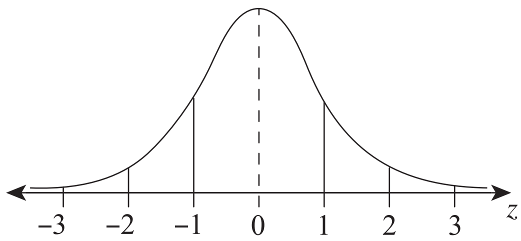
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A | B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \int_a^r f(x)dx$

$P(a < X < b) = \int_a^b f(x)dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow P(X = x)$

$= \binom{n}{x}p^x(1 - p)^{n-x}, x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \ldots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \ldots + \binom{n}{r}x^{n-r}a^{r} + \ldots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2} + y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$