ACE
EXAM
PAPER

Student name: ________________

2022

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 1

General Instructions

- Working time - 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt all questions
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Given $f(x) = \sqrt{x} - 3$, what are the domain and range of $f^{-1}(x)$?

(A) $x \geq -3,\ \ y \geq 0$

(B) $x \geq -3,\ \ y \geq -3$

(C) $x \geq 0,\ \ y \geq 0$

(D) $x \geq 0,\ \ y \geq -3$

2. What is the value of $\sin 2x$ given that $\sin x = 0.8$ and x is obtuse ?

(A) $-\frac{12}{25}$

(B) $-\frac{24}{25}$

(C) $\frac{12}{25}$

(D) $\frac{24}{25}$

3. Jack starts at the origin and walks along vector $2\underset{\sim}{i} + 3\underset{\sim}{j}$ and then turns and walks along vector $4\underset{\sim}{i} - 2\underset{\sim}{j}$. How far is Jack from the origin ?

(A) 5

(B) $\sqrt{11}$

(C) $\sqrt{37}$

(D) $\sqrt{61}$

4. What is the derivative of $f(x) = \tan^{-1}\frac{1}{x}$?

(A) $\frac{-1}{1+x^2}$

(B) $\frac{-x^2}{1+x^2}$

(C) $\frac{1}{1+x^2}$

(D) $\frac{x^2}{1+x^2}$

5. Layla projects an arrow at an angle of 45° to the horizontal with an initial velocity of 50 ms^{-1}. What is the horizontal speed of the arrow ?

(A) $\sqrt{2}$ ms^{-1}

(B) $50\sqrt{2}$ ms^{-1}

(C) $\frac{1}{\sqrt{2}}$ ms^{-1}

(D) $\frac{50}{\sqrt{2}}$ ms^{-1}

6. A school committee consists of 8 members and a chairperson. The members are selected from 12 students. The chairperson is selected from 4 teachers. In how many ways could the committee be selected ?

(A) ${}^{12}C_8 + {}^{4}C_1$

(B) ${}^{12}P_8 + {}^{4}P_1$

(C) ${}^{12}P_8 \times {}^{4}P_1$

(D) ${}^{12}C_8 \times {}^{4}C_1$

7. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

What are the values of p for which $y = e^{px}$ satisfies the above differential equation ?

(A) $p = -4,\ p = 3$

(B) $p = -3,\ p = 4$

(C) $p = \pm 3$

(D) $p = \pm 4$

8. What is the correct expression for the indefinite integral $\int (4\cos^2 x + \sec^2 x)dx$?

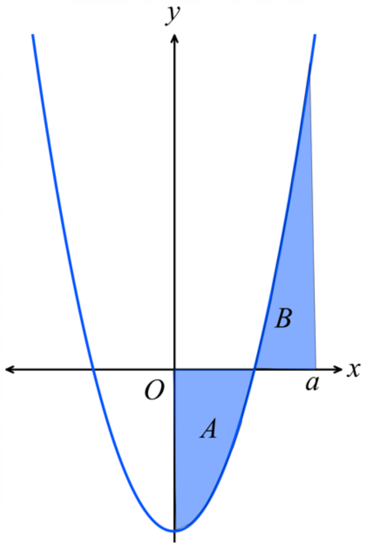
(A) $2x + 2\sin 2x + \tan x + C$

(B) $2x - 2\sin 2x + \tan x + C$

(C) $2x + \sin 2x + \tan x + C$

(D) $2x - \sin 2x + \tan x + C$

9. The graph of $y = x^2 - 4$ s shown below.



Not to scale

The area of the region A is the equal to the area of the region B. What is the value of a ?

(A) 6

(B) $\sqrt{6}$

(C) $2\sqrt{3}$

(D) 12

10. Harry knows that each ticket has a probability of 0.15 of winning a prize in a lucky ticket competition. He buys 30 tickets. What is a general rule for the probability distribution of the number of winning tickets?

(A) $P(X = x) = {}^{15}C_x\, 0.30^x 0.70^{15-x}$

(B) $P(X = x) = {}^{15}C_x\, 0.30^x 0.70^{15-x}$

(C) $P(X = x) = {}^{30}C_x\, 0.15^x 0.85^{30-x}$

(D) $P(X = x) = {}^{30}C_x\, 0.85^x 0.15^{30-x}$

Section II

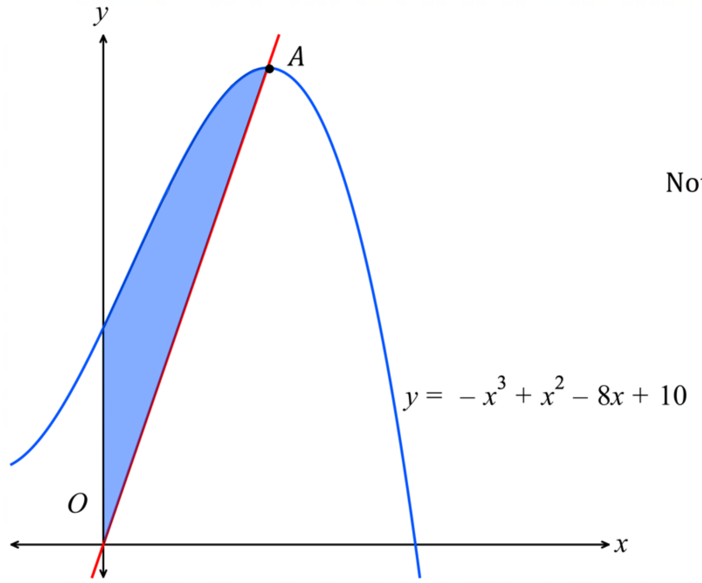
60 marks
Attempt all questions
Allow about 1 hour and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) An examination consists of 40 multiple-choice questions, each question having five possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is $E(X)$? **2**

(b) A curve with the equation $y = -x^3 + x^2 + 8x + 10$ has a maximum turning point at A. The shaded region shown below, is bounded by the curve, the y-axis and the line from O to A, where O is the origin.



Not to scale

(i) Show that the x-coordinate of A is 2. **1**

(ii) Find the area of the region. **3**

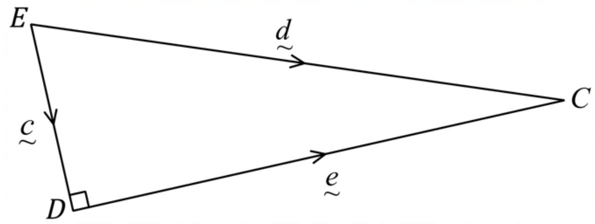
(c) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\,dx.$ **2**

(d) (i) Find the remainder when $x^3 - 2x^2 - 4x + 8$ is divided by $x - 2$. **1**

(ii) Hence or otherwise find all the solutions to the equation: **2**

$x^3 - 2x^2 - 4x + 8 = 0$

(e) ΔDEC has a right angle at D.



Show that :

(i) $|\underset{\sim}{d}|^2 = \underset{\sim}{e} \cdot \underset{\sim}{e} + 2(\underset{\sim}{e} \cdot \underset{\sim}{c}) + \underset{\sim}{c} \cdot \underset{\sim}{c}$ **2**

(ii) $|d|^2 = |\underset{\sim}{e}|^2 + |c|^2$ **2**

Question 12 (15 marks) **Marks**

(a) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x dx$. **3**

(b) Solve $\frac{dy}{dx} = e^{3y}$ by finding x as a function of y . **2**

(c) A missile is shot from the origin O with initial speed of 64 ms^{-1} at an angle of 30° to the horizontal. The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -10$.

(i) Show that $x = 32\sqrt{3}t$. **1**

(ii) Show that $y = 32t - 5t^2$. **2**

(iii) What is the equation of the trajectory of the missile ? **1**

(d) A multiple-choice test contains ten questions. Each question has four choices for the correct answer. Only one of the choices is correct. A student guesses the answers to all the questions.

(i) What is the probability of getting all the questions correct ? **1**

(ii) What is the probability of getting at most 90% ? **1**

(iii) What is the probability of getting over 75% ? **2**

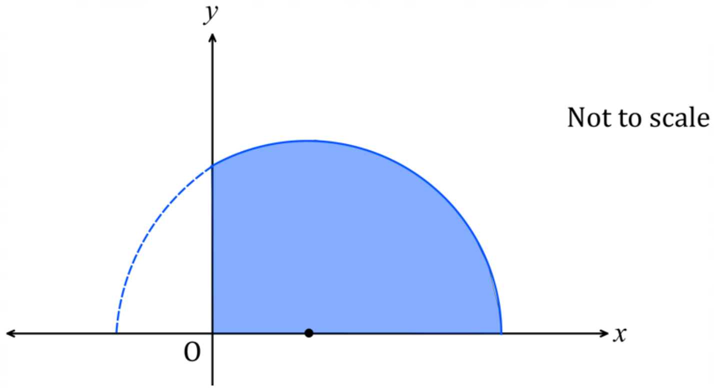
(e) Find the angle between the vectors $\underset{\sim}{a} = -2\underset{\sim}{i} + 6\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$. **2**

Question 13 (15 marks) **Marks**

(a) Given that $y = e^{2x} + e^{-2x}$, determine the values of constants a and b that satisfy the following equation: **3**

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 5e^{2x} + e^{-2x}$$

(b) **3**



A semi-circle with centre (1, 0) and radius 2, lies on the x-axis as shown above. Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.

(c) Using the substitution $u = 4 - x$ or otherwise, find $\int 3x\sqrt{4-x}\,dx$. **3**

(d) Prove by mathematical induction that: **3**

$n^3 + 2n$ is divisible by 3 for all positive integers n $(n \geq 1)$.

(e) Use the binomial theorem to expand $\left(\frac{1}{2}x - 3\right)^4$. Simplify your answer. **3**

Question 14 (15 marks) **Marks**

(a) Use the principle of mathematical induction to prove that: **3**

$$1 + 4 + 16 + \ldots + 4^n = \frac{1}{3}[4^{n+1} - 1]$$

where n is a positive integer greater than or equal to zero.

(b) (i) Prove the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Hence find expressions for the exact values of the solutions to the equation $8a^3 - 6a = 1$. **3**

(c) A population of platypus has an initial population of 200. Birth rates and the amount of food affect the population of the platypus. The decrease in the population, *P*, is given by the formula: **3**

$$P = \frac{200}{1 + 500e^{-1\cdot 5t}}$$ where k is a constant and t is in months

How long will it take for only 40 platypus to remain?
Give your answer to the nearest month.

(d) (i) Show that $\frac{\sec^2 x}{\tan x} = \frac{\operatorname{cosec} x}{\cos x}$ **2**

(ii) Use the substitution $u = \tan x$ to find the exact value of the integral: **2**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\operatorname{cosec} x}{\cos x}\,dx$$

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Financial Mathematics

$A = P(1+r)^n$

Sequences and series

$T_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \ r \neq 1$

$S = \frac{a}{1-r}, \ |r| < 1$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x-h)^2 + (y-k)^2 = r^2$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

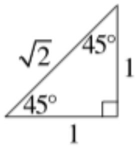
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

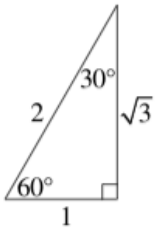
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

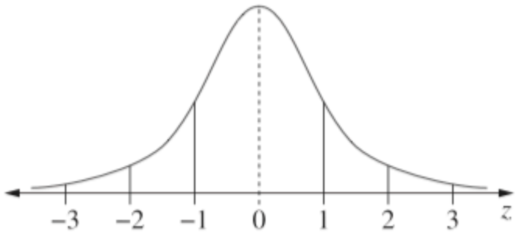
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$