NSW GOVERNMENT

NSW Education Standards Authority

2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number on the Question 12 Writing Booklet attached

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–12)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given that $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, what is $\overrightarrow{PQ}$?

A. $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$

B. $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

D. $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$

2 Which of the following integrals is equivalent to $\int \sin^2 3x \, dx$?

A. $\displaystyle\int \frac{1 + \cos 6x}{2} \, dx$

B. $\displaystyle\int \frac{1 - \cos 6x}{2} \, dx$

C. $\displaystyle\int \frac{1 + \sin 6x}{2} \, dx$

D. $\displaystyle\int \frac{1 - \sin 6x}{2} \, dx$

3 What is the remainder when $P(x) = -x^3 - 2x^2 - 3x + 8$ is divided by $x + 2$?

A. -14

B. -2

C. 2

D. 14

4 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$.

Which of the following equations best represents this relationship between x and y?

A. $y^2 = x^2 + c$

B. $y^2 = \dfrac{x^2}{2} + c$

C. $y = x \ln|y| + c$

D. $y = \dfrac{x^2}{2} \ln|y| + c$

5 For the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ it is known that

$$\overrightarrow{OA} \cdot \overrightarrow{OB} < 0.$$

Which of the following statements MUST be true?

A. Either, $\overrightarrow{OA}$ is negative and $\overrightarrow{OB}$ is positive, or, $\overrightarrow{OA}$ is positive and $\overrightarrow{OB}$ is negative.

B. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is obtuse.

C. The product $\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|$ is negative.

D. The points O, A and B are collinear.

6 The random variable X represents the number of successes in 10 independent Bernoulli trials. The probability of success is $p = 0.9$ in each trial.

Let $r = P(X \geq 1)$.

Which of the following describes the value of r?

A. $r > 0.9$

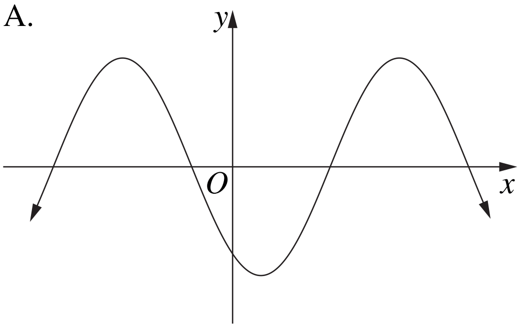
B. $r = 0.9$

C. $0.1 < r < 0.9$

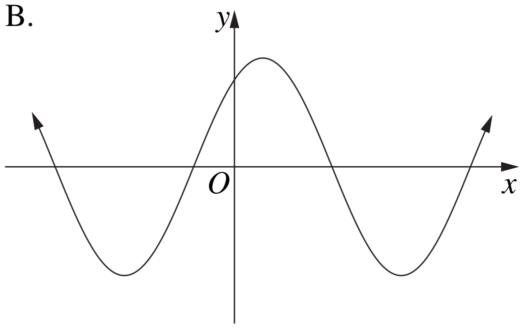
D. $r \leq 0.1$

7 Which curve best represents the graph of the function $f(x) = -a\sin x + b\cos x$ given that the constants a and b are both positive?

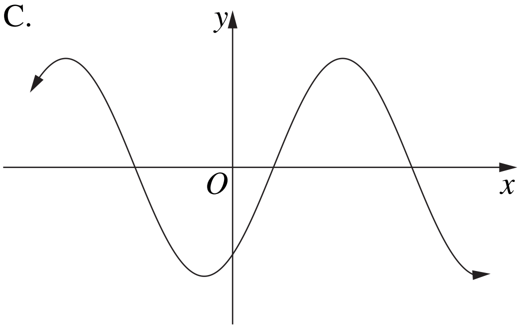
A.



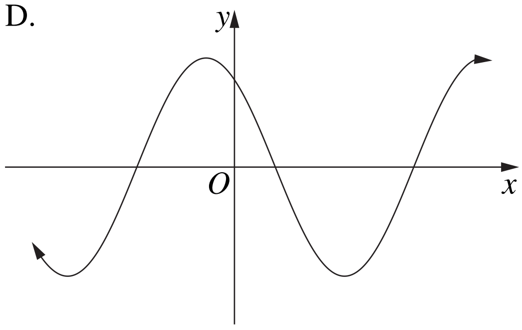
B.



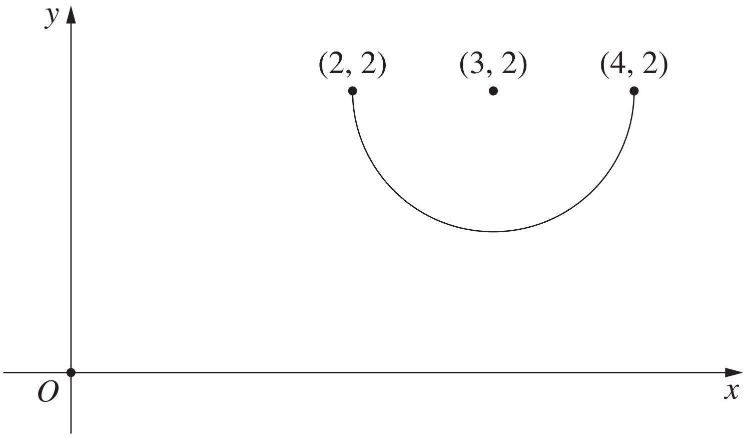
C.



D.



8 The diagram shows a semicircle.



Which pair of parametric equations represents the semicircle shown?

A. $\begin{cases} x = 3 + \sin t \\ y = 2 + \cos t \end{cases}$ for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

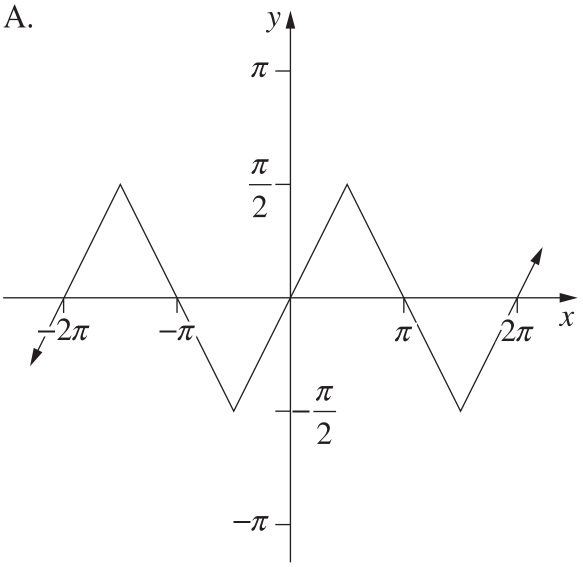
B. $\begin{cases} x = 3 + \cos t \\ y = 2 + \sin t \end{cases}$ for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

C. $\begin{cases} x = 3 - \sin t \\ y = 2 - \cos t \end{cases}$ for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

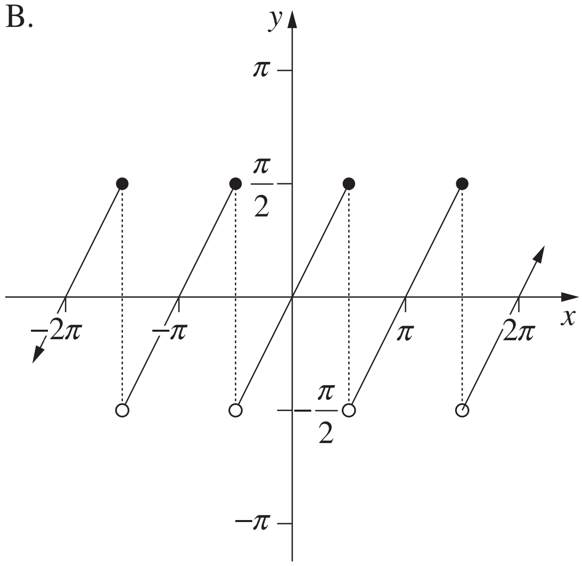
D. $\begin{cases} x = 3 - \cos t \\ y = 2 - \sin t \end{cases}$ for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

9 Which graph represents the function $y = \sin^{-1}(\sin x)$?

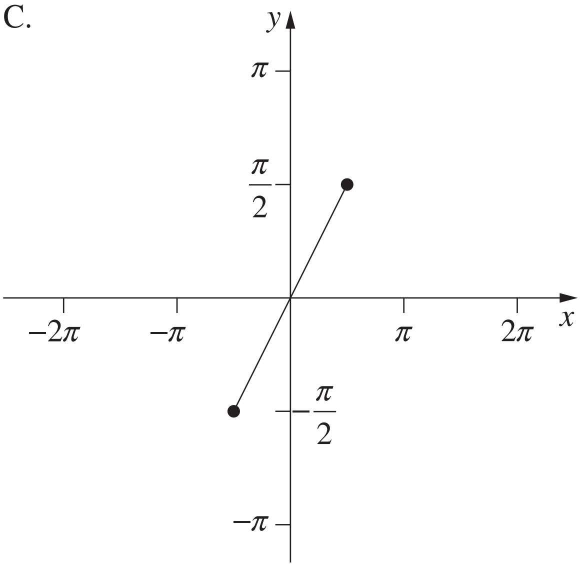
A.



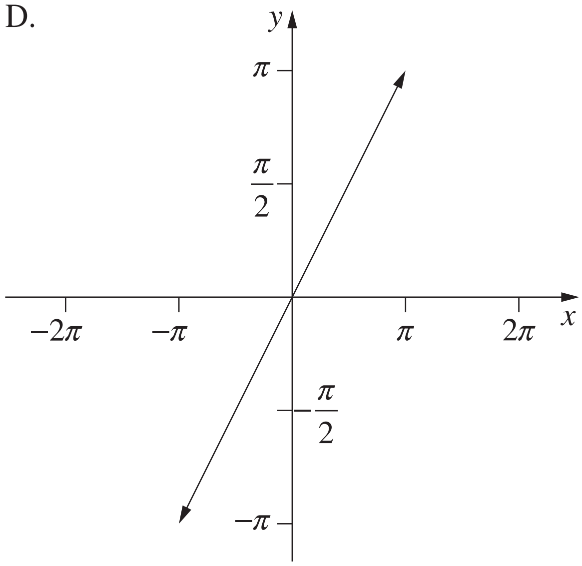
B.



C.



D.



10 The members of a club voted for a new president. There were 15 candidates for the position of president and 3543 members voted. Each member voted for one candidate only.

One candidate received more votes than anyone else and so became the new president.

What is the smallest number of votes the new president could have received?

A. 236

B. 237

C. 238

D. 239

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Find $\left(\underset{\sim}{i} + 6\underset{\sim}{j}\right) + \left(2\underset{\sim}{i} - 7\underset{\sim}{j}\right)$. **1**

(b) Expand and simplify $(2a - b)^4$. **2**

(c) Use the substitution $u = x + 1$ to find $\displaystyle\int x\sqrt{x+1}\,dx$. **3**

(d) A committee containing 5 men and 3 women is to be formed from a group of 10 men and 8 women. **1**

In how many different ways can the committee be formed?

(e) A spherical bubble is moving up through a liquid. As it rises, the bubble gets bigger and its radius increases at the rate of 0.2 mm/s. **2**

At what rate is the volume of the bubble increasing when its radius reaches 0.6 mm? Express your answer in mm^3/s rounded to one decimal place.

(f) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\,dx$. **2**

(g) By factorising, or otherwise, solve $2\sin^3 x + 2\sin^2 x - \sin x - 1 = 0$ for $0 \le x \le 2\pi$. **3**

(h) The roots of $x^4 - 3x + 6 = 0$ are α, β, γ and δ. **2**

What is the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$?

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) The direction field for a differential equation is given on page 1 of the Question 12 Writing Booklet. **1**

The graph of a particular solution to the differential equation passes through the point P.

On the diagram provided in the writing booklet, sketch the graph of this particular solution.

(b) A bottle of water, with temperature 5°C, is placed on a table in a room. The temperature of the room remains constant at 25°C. After t minutes, the temperature of the water, in degrees Celsius, is T.

The temperature of the water can be modelled using the differential equation

$$\frac{dT}{dt} = k(T - 25) \qquad \text{(Do NOT prove this.)}$$

where k is the growth constant.

(i) After 8 minutes, the temperature of the water is 10°C. **3**

By solving the differential equation, find the value of t when the temperature of the water reaches 20°C. Give your answer to the nearest minute.

(ii) Sketch the graph of T as a function of t. **1**

(c) Use mathematical induction to prove that **3**

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

for all integers $n \geq 1$.

(d) A function is defined by $f(x) = 4 - \left(1 - \frac{x}{2}\right)^2$ for x in the domain $(-\infty, 2]$.

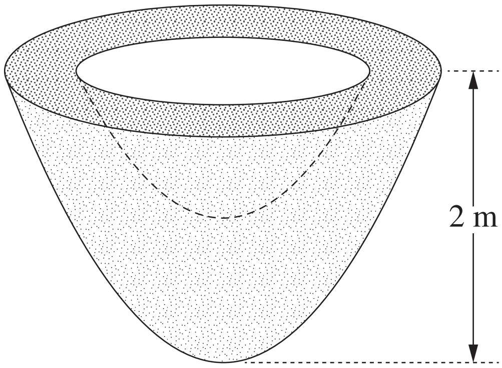
(i) Sketch the graph of $y = f(x)$ showing the x- and y-intercepts. **2**

(ii) Find the equation of the inverse function, $f^{-1}(x)$, and state its domain. **3**

(iii) Sketch the graph of $y = f^{-1}(x)$. **1**

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) A 2-metre-high sculpture is to be made out of concrete. The sculpture is formed by rotating the region between $y = x^2$, $y = x^2 + 1$ and $y = 2$ around the y-axis. **3**



Find the volume of concrete needed to make the sculpture.

(b) When an object is projected from a point h metres above the origin with initial speed V m/s at an angle of $\theta°$ to the horizontal, its displacement vector, t seconds after projection, is **4**

$$\underset{\sim}{r}(t) = \left(Vt\cos\theta\right)\underset{\sim}{i} + \left(-5t^2 + Vt\sin\theta + h\right)\underset{\sim}{j}.$$

(Do NOT prove this.)

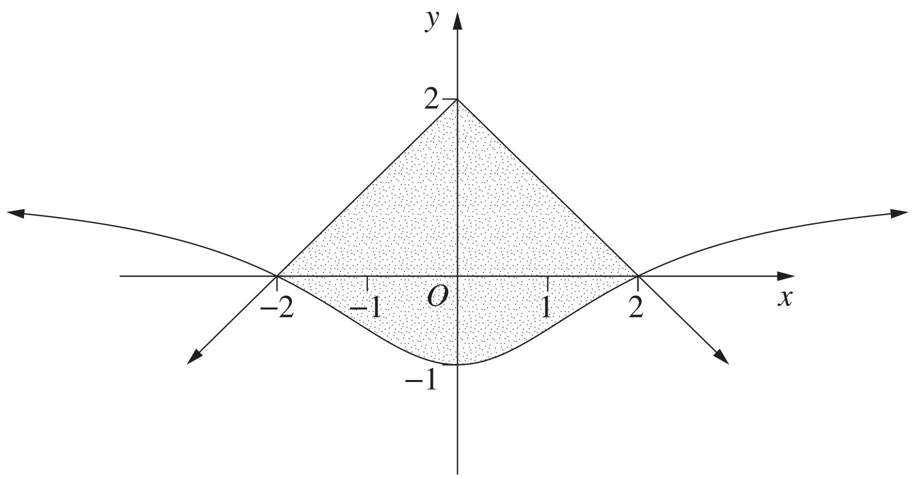
A person, standing in an empty room which is 3 m high, throws a ball at the far wall of the room. The ball leaves their hand 1 m above the floor and 10 m from the far wall. The initial velocity of the ball is 12 m/s at an angle of 30° to the horizontal.

Show that the ball will NOT hit the ceiling of the room but that it will hit the far wall without hitting the floor.

Question 13 continues on page 10

Question 13 (continued)

(c) The region enclosed by $y = 2 - |x|$ and $y = 1 - \dfrac{8}{4 + x^2}$ is shaded in the diagram. **3**



Find the exact value of the area of the shaded region.

(d) (i) The numbers A, B and C are related by the equations $A = B - d$ and $C = B + d$, where d is a constant. **2**

Show that $\dfrac{\sin A + \sin C}{\cos A + \cos C} = \tan B$.

(ii) Hence, or otherwise, solve $\dfrac{\sin\frac{5\theta}{7} + \sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7} + \cos\frac{6\theta}{7}} = \sqrt{3}$, for $0 \le \theta \le 2\pi$. **2**

End of Question 13

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) A plane needs to travel to a destination that is on a bearing of 063°. The engine is set to fly at a constant 175 km/h. However, there is a wind from the south with a constant speed of 42 km/h. **3**

On what constant bearing, to the nearest degree, should the direction of the plane be set in order to reach the destination?

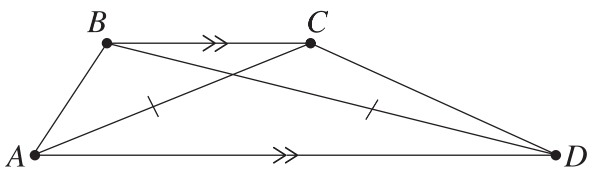
(b) In a certain country, the population of deer was estimated in 1980 to be 150 000. The population growth is given by the logistic equation $\frac{dP}{dt} = 0.1P\left(\frac{C-P}{C}\right)$ where t is the number of years after 1980 and C is the carrying capacity. **4**

In the year 2000, the population of deer was estimated to be 600 000.

Use the fact that $\frac{C}{P(C-P)} = \frac{1}{P} + \frac{1}{C-P}$ to show that the carrying capacity is approximately 1 130 000.

(c) (i) For vector $\underset{\sim}{v}$, show that $\underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2$. **1**

(ii) In the trapezium $ABCD$, BC is parallel to AD and $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$. **3**



NOT TO SCALE

Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\overrightarrow{BC}$, where $k > 0$.

Using part (i), or otherwise, show $2\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)\,|\underset{\sim}{b}|^2 = 0$.

Question 14 continues on page 12

(d) At a certain factory, the proportion of faulty items produced by a machine is $p = \dfrac{3}{500}$, which is considered to be acceptable. To confirm that the machine is working to this standard, a sample of size n is taken and the sample proportion $\hat{p}$ is calculated. **3**

It is assumed that $\hat{p}$ is approximately normally distributed with $\mu = p$ and $\sigma^2 = \dfrac{p(1-p)}{n}$.

Production by this machine will be shut down if $\hat{p} \geq \dfrac{4}{500}$.

The sample size is to be chosen so that the chance of shutting down the machine unnecessarily is less than 2.5%.

Find the approximate sample size required, giving your answer to the nearest thousand.

(e) The polynomial $g(x) = x^3 + 4x - 2$ passes through the point $(1, 3)$. **2**

Find the gradient of the tangent to $f(x) = xg^{-1}(x)$ at the point where $x = 3$.

End of paper

Centre Number

Student Number

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2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Writing Booklet

Question 12

Instructions

- Use this Writing Booklet to answer Question 12.
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3).
- Write your Centre Number and Student Number at the top of this page.
- Write using black pen.
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

12

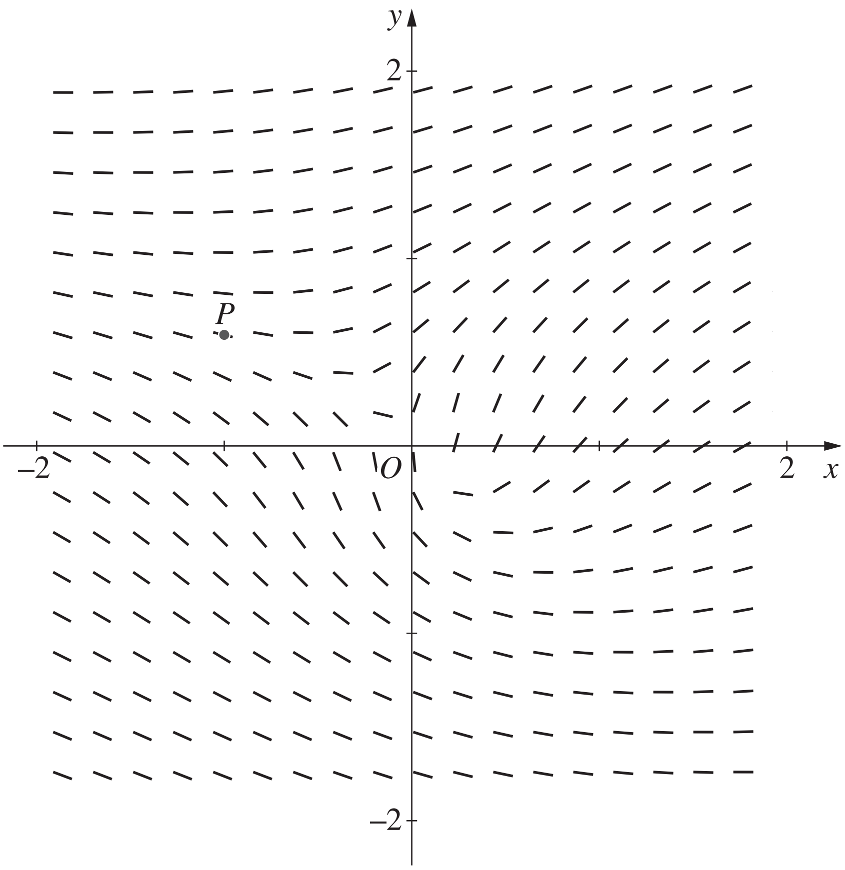
of

this booklet

number of booklets for this question

Start here for Question Number: **12**

(a) Sketch the graph of the particular solution that passes through the point P.



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Additional writing space on back page.

Tick this box if you have continued this answer in another writing booklet.

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2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

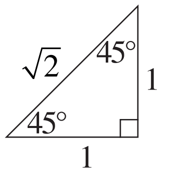
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

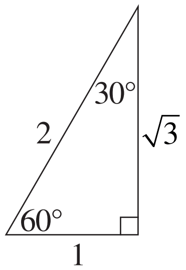


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

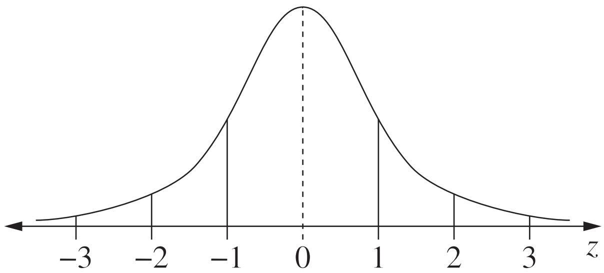
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

$$\binom{n}{r}={}^{n}C_{r}=\frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n=x^n+\binom{n}{1}x^{n-1}a+\cdots+\binom{n}{r}x^{n-r}a^r+\cdots+a^n$$

Vectors

$$\left|\underset{\sim}{u}\right|=\left|x\underset{\sim}{i}+y\underset{\sim}{j}\right|=\sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v}=\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta=x_1x_2+y_1y_2,$$

where $\underset{\sim}{u}=x_1\underset{\sim}{i}+y_1\underset{\sim}{j}$

and $\underset{\sim}{v}=x_2\underset{\sim}{i}+y_2\underset{\sim}{j}$

$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned}z=a+ib&=r(\cos\theta+i\sin\theta)\\&=re^{i\theta}\end{aligned}$$

$$\begin{aligned}\left[r(\cos\theta+i\sin\theta)\right]^n&=r^n(\cos n\theta+i\sin n\theta)\\&=r^ne^{in\theta}\end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x=a\cos(nt+\alpha)+c$$

$$x=a\sin(nt+\alpha)+c$$

$$\ddot{x}=-n^2(x-c)$$