Fort Street High School

2021 ASSESSMENT TASK 4 - ONLINE EXAMINATION

Mathematics Advanced

Total Marks: 55 Working time: 95 minutes

General Instructions

- This is an open book task
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet will be provided
- Show relevant mathematical reasoning and/or calculations

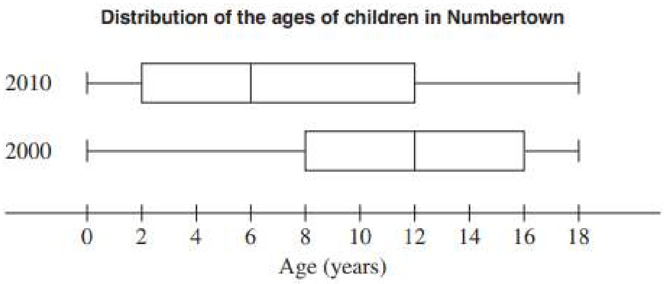
Specific Instructions:

- You will not be able to leave your desk for the duration of the task.
- Mobile phones must be turned off and out of sight.
- Your microphone and cameras must be on, but you can turn the volume on your devices down so that any noise from other students does not disturb you.
- You are not permitted to have headphones/ear buds and if you have long hair, please tie it back.
- You can ask for assistance through the direct chat function of Zoom/Teams or ask as your microphone is on.
- You are to manage your time and make sure you have a timer at hand to keep within the assessment time limit.
- At the end of the assessment, you will have 15 minutes to scan and submit your task on Google Classroom. During this time, if you have any difficulty with submitting your task, please communicate this with your teacher immediately.
- Please note that submission of the task is your responsibility.

Attempt Questions 1-6
Answer each question in a new writing booklet.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (11 marks) Start a new writing booklet **Marks**

(a) The graphs below show the distribution of the ages of children in Numbertown in 2000 and 2010. In the year 2010, there were 2500 children. The number of children aged 12-18 is the same in 2000 and 2010.



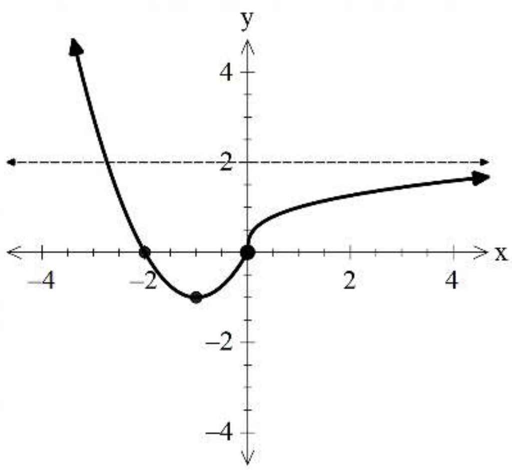
(i) Describe the distribution of the ages of children in 2000 by making explicit reference to the measure of spread, location, and skew. **3**

(ii) How many children were aged 0-2 in 2010? **1**

(iii) How many children aged 0-18 were there in 2000? **1**

(b) Determine $f(x)$ given that $f'(x) = \sec^2 2x + \sin 2x$ and $f(0) = 1$. **3**

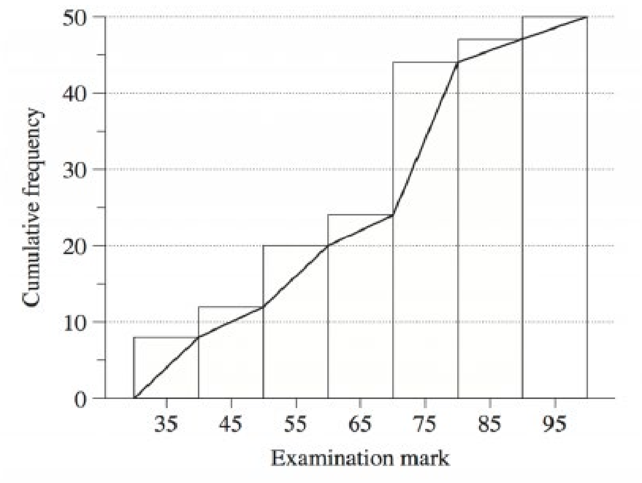
(c) The curve of $y = f(x)$ is given below. Sketch $y = -2f(-2x)$, indicating all key features of your sketch including intercepts, turning points and any asymptotes. The dashed line represents an asymptote and there is a turning point at (-1, -1). **3**



End of Question 1

Question 2 (10 marks) Start a new writing booklet **Marks**

(a) A set of examination results is displayed in a cumulative frequency histogram and polygon (ogive). The marks have been sorted into groups and the marks given on the diagram are class centres.



(i) Find the median examination mark. **1**

(ii) Estimate the percentage of students that achieved a mark below 45. **1**

(iii) A student knows that their mark is between the 6^{th} and the 7^{th} decile. Give a possible examination mark for the student. **1**

(b) A particle is moving so that its velocity in m/s is given by $v = 2 - 2\sin(2\pi t)$ where t is in seconds. The particle is initially at the origin.

(i) Explain why the particle never changes direction. **1**

(ii) Find the displacement x of the particle after 1 second. **3**

(iii) For sufficiently large values of t (as $t \to \infty$), describe the behaviour of the displacement of the particle. **1**

(iv) Find the exact acceleration of the particle at $t = 4$ seconds. **2**

End of Question 2

Question 3 (8 marks) Start a new writing booklet **Marks**

(a) (i) Show that $\frac{d}{dx}\left(\ln\left(\frac{3+x}{3-x}\right)\right)=\frac{6}{9-x^2}$. **3**

(ii) Hence find $\int\frac{dx}{9-x^2}$. **1**

(b) Find the exact area bounded by the curve $y=x^2-2$, the x axis and the line $y=4$. **4**

End of Question 3

Question 4 (9 marks) Start a new writing booklet **Marks**

(a) The first term of a geometric series is e^x and the fifth term is $81e^{9x}$.

(i) Show that the common ratio can be expressed by $r = 3e^{2x}$. **2**

(ii) Find an expression for the 8^{th} term of the series. **1**

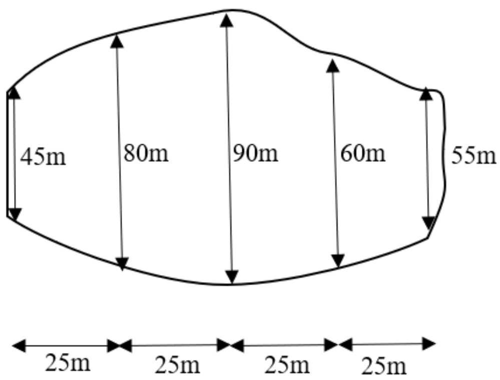
(iii) For what values of x does the series have a limiting sum? **3**

(iv) Find the exact value of x if the limiting sum is 3. **3**

End of Question 4

Question 5 (7 marks) Start a new writing booklet **Marks**

(a) A factory located next to a lake has recently been shut down due to concerns about chemical run off. A bird's eye view of the lake is given in the diagram below.



Local volunteers have begun a clean-up effort to remove these harmful chemicals from the lake. The concentration of chemicals is given by the equation:

$$C = 1.5 - 0.4e^{kt}$$

Let C be the concentration of chemicals in the lake in kg/m^3, where k is a constant and t is the number of years after the clean-up effort has started.

(i) The initial weight of chemical run off in the lake is 38500kg, and the average depth of the lake is 5m. Use the trapezoidal rule to show that the initial concentration of chemical run off can be estimated as 1.1 kg/m^3. **3**

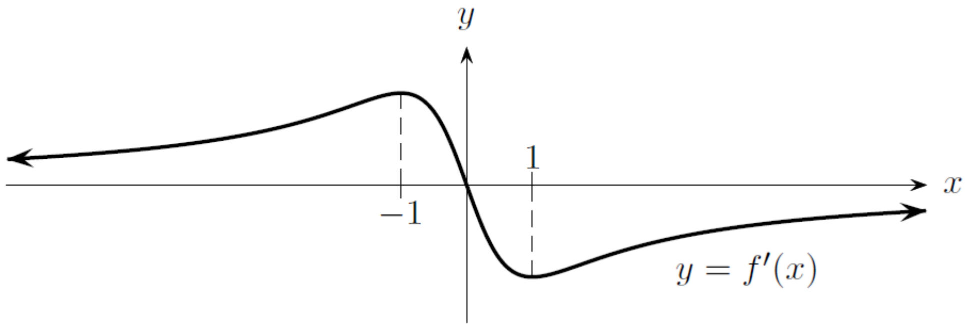
(ii) Find the value of k to three significant figures if it takes 2.45 years to remove all chemical run off from the lake **2**

(iii) Find the rate of change in the concentration of chemical run off in the lake at $t = 1.8$ years. Round your answer to two decimal places. **2**

End of Question 5

Question 6 (10 marks) Start a new writing booklet **Marks**

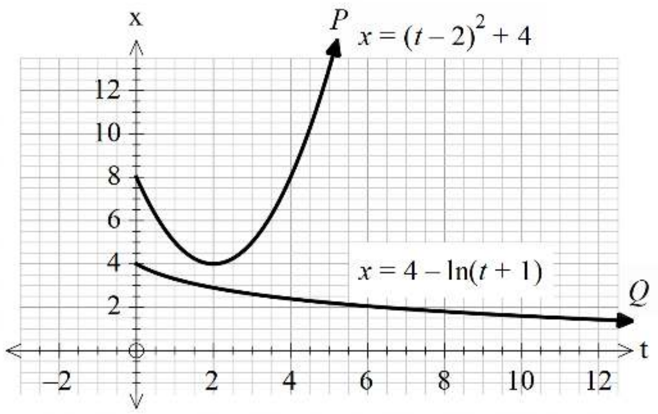
(a) The graph of $y = f'(x)$ is shown in the diagram below. $y = f'(x)$ passes through the origin. As $x \to \pm\infty$, $f'(x) \to 0$ and $f(x) \to 0$. **3**



Sketch the graph of $y = f(x)$, given $f(x) > 0$ for all values of x.

(b) Two particles P and Q start moving along the x axis at time $t = 0$. Particle P is initially at $x = 8$ and its velocity v in m/s at time t in seconds is given by $v = 2t - 4$.

The position of particle Q is given by $x = 4 - \ln(t+1)$. The diagram shows the graph of $x = 4 - \ln(t+1)$.



(i) Show that the position of particle P is given by $x = (t-2)^2 + 4$. **2**

(ii) Explain why particles P and Q will never collide. **1**

(iii) Show that the distance between the particles can be given by $PQ = (t-2)^2 + \ln(t+1)$. **1**

(iv) Find the exact time that the distance between the particles PQ is a minimum. **3**

End of Examination

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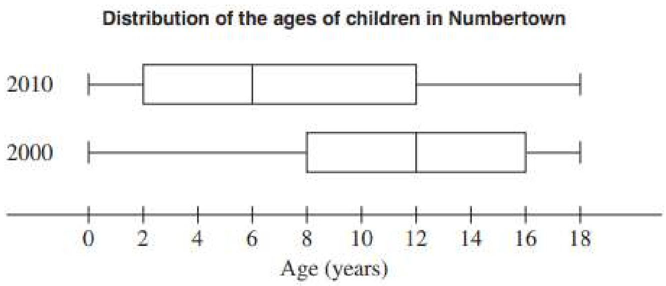


(i) Describe the distribution of the ages of children in 2000 by making explicit reference to the measure of spread, location, and skew. **3**

(ii) How many children were aged 0-2 in 2010? **1**

(iii) How many children aged 0-18 were there in 2000? **1**

i) The distribution of children's ages in 2000 is negatively skewed and has a median of 12, interquartile range of 8 and range of 18.

ii) $0.25(2500) = 625$ children

iii) In 2010 there were 625 children aged 12-18, and this represented half of the population.
In 2000 there is the same number of children aged 12-18, but this represents half of the data
This means there is 1250 children in 2000.

i) 1 Mark for correct mention of skew 1 mark for correct median 1 mark for mention of IQR or Range **ii)** 1 mark for correct answer **iii)** 1 mark for correct answer

Markers comments:

(b) Determine $f(x)$ given that $f'(x) = \sec^2 2x + \sin 2x$ and $f(0) = 1$. **3**

$f'(x) = \sec^2(2x) + \sin(2x)$

$f(x) = \frac{\tan(2x)}{2} - \frac{\cos(2x)}{2} + C$

$f(0) = \frac{\tan(0)}{2} - \frac{\cos(0)}{2} + C = 1$

$0 - \frac{1}{2} + C = 1$

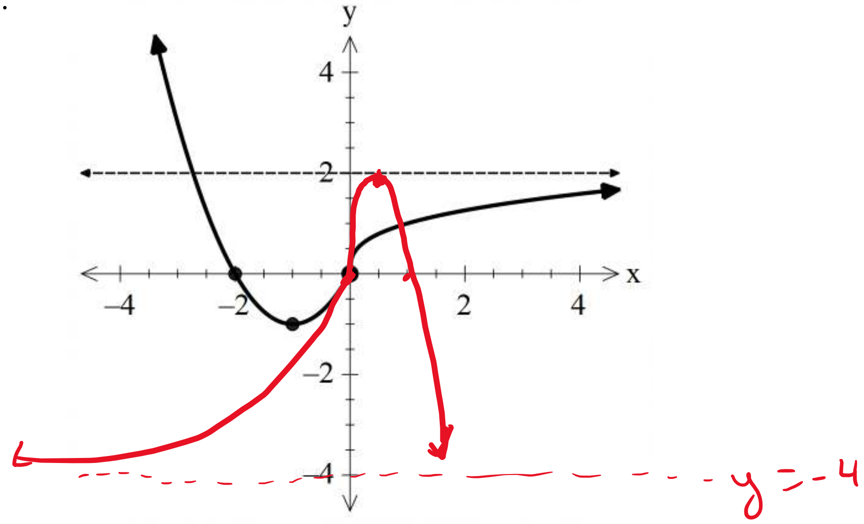
$C = \frac{3}{2}$

so $f(x) = \frac{1}{2}(\tan(2x) - \cos(2x) + 3)$

1 mark for correct integration 1 mark for finding C 1 mark for final answer

Markers comments:

(c) The curve of $y = f(x)$ is given below. Sketch $y = -2f(-2x)$, indicating all key features of your sketch including intercepts, turning points and any asymptotes. The dashed line represents an asymptote and there is a turning point at (-1, -1). **3**



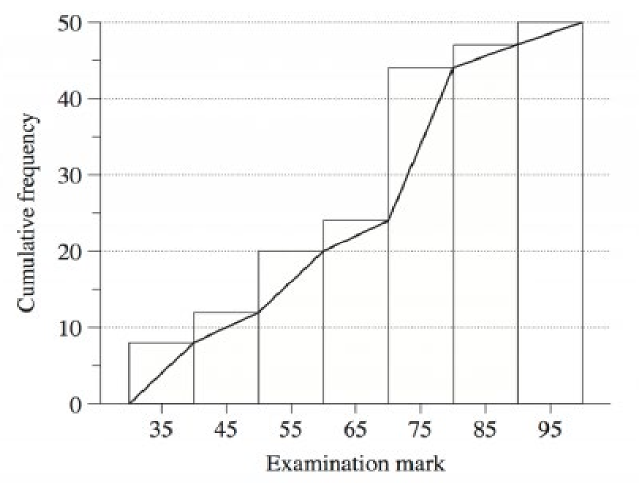
1 mark for correct reflections 1 mark for correct vertical dilation 1 mark for correct horizontal dilation

Markers comments:

End of Question 1

Question 2 (10 marks) Start a new writing booklet **Marks**

(a) A set of examination results is displayed in a cumulative frequency histogram and polygon (ogive). The marks have been sorted into groups and the marks given on the diagram are class centres.



(i) Find the median examination mark. **1**

(ii) Estimate the percentage of students that achieved a mark below 45. **1**

(iii) A student knows that their mark is between the 6th and the 7th decile. Give a possible examination mark for the student. **1**

i) 75

ii) 20%

iii) 75 (anything in the 70's is acceptable)

i) 1 mark for correct answer

ii) 1 mark for correct answer

iii) 1 mark for correct answer

Markers comments:

(b) A particle is moving so that its velocity in m/s is given by $v = 2 - 2\sin(2\pi t)$ where t is in seconds. The particle is initially at the origin.

(i) Explain why the particle never changes direction. **1**

(ii) Find the displacement x of the particle after 1 second. **3**

(iii) For sufficiently large values of t (as $t \to \infty$), describe the behaviour of the displacement of the particle. **1**

(iv) Find the exact acceleration of the particle at $t = 4$ seconds. **2**

i) The range of the velocity equation is [0, 4], so the $v \geq 0$ for all t. Therefore the particle is only ever stationary or travelling in the positive direction.

ii) $v = 2 - 2\sin(2\pi t)$

$x = 2t + \frac{1}{\pi}\cos(2\pi t) + c$

when $t = 0$, $x = 0$

$x = 2(0) + \frac{1}{\pi}\cos(2\pi(0)) + c$

$= 0 + \frac{1}{\pi} + c$

$c = -\frac{1}{\pi}$

so $x = 2t + \frac{1}{\pi}\cos(2\pi t) - \frac{1}{\pi}$

when $t = 1$, $x = 2(1) + \frac{1}{\pi}\cos(2\pi) - \frac{1}{\pi}$

$= 2 + \frac{1}{\pi} - \frac{1}{\pi}$

$= 2\text{ m}$

iii) * $x = 2t + \frac{1}{\pi}\cos(2\pi t) - \frac{1}{\pi}$

as $t \to \infty$, $x \to \infty$ so x approaches infinity.

* Students can also use the fact that the particle never changes direction from part (b) to answer this question.

IV) $v = 2 - 2\sin(2\pi t)$

$a = -4\pi\cos(2\pi t)$

when $t = 4$

$a = -4\pi(\cos 8\pi)$

$a = -4\pi\ \text{ms}^{-2}$

i) 1 mark for correct answer

ii) 1 mark for correct integration

1 mark for finding C

1 mark for finding correct function

iii) 1 mark for correct answer

iv) 1 mark for correct acceleration equation

1 mark for correct value

Markers comments:

Question 3 (8 marks) Start a new writing booklet **Marks**

(a) (i) Show that $\frac{d}{dx}\left(\ln\left(\frac{3+x}{3-x}\right)\right)=\frac{6}{9-x^2}$. **3**

(ii) Hence find $\int \frac{dx}{9-x^2}$. **1**

i) $\frac{d}{dx}\left[\ln\left(\frac{3+x}{3-x}\right)\right] = \dfrac{\left[\frac{(3-x)(1)-(3+x)(-1)}{(3-x)^2}\right]}{\left[\frac{3+x}{3-x}\right]}$

$= \left(\frac{3-\cancel{x}+3+\cancel{x}}{(3-x)^{\cancel{2}}}\right)\left(\frac{\cancel{3-x}}{3+x}\right)$

$= \frac{6}{(3-x)(3+x)}$

$= \frac{6}{9-x^2}$

ii) Hence if $\int \frac{6}{9-x^2}\,dx = \ln\left(\frac{3+x}{3-x}\right)+C$, then

$\int \frac{1}{9-x^2}\,dx = \frac{1}{6}\int\frac{6}{9-x^2}\,dx$

$= \frac{1}{6}\ln\left(\frac{3+x}{3-x}\right)+C$

i) 1 mark for using quotient rule or log laws

1 mark for correctly applying integration rule

1 mark for correct answer

ii) 1 mark for correct answer

Markers comments:

i) alternative

$\frac{d}{dx}\left(\ln\left(\frac{3+x}{3-x}\right)\right) = \frac{d}{dx}\left[\ln(3+x)-\ln(3-x)\right]$

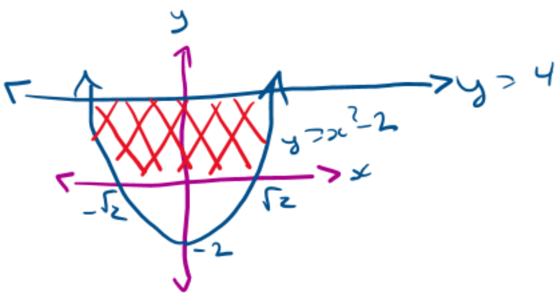
$= \frac{1}{3+x}+\frac{1}{3-x}$

$= \frac{3-x+(3+x)}{(3+x)(3-x)}$

$= \frac{6}{(3+x)(3-x)}$

$= \frac{6}{9-x^2}$

(b) Find the exact area bounded by the curve $y = x^2 - 2$, the x axis and the line $y = 4$. **4**



$y = x^2 - 2$

$x = \pm\sqrt{y+2}$

$A = 2\int_0^4 \sqrt{y+2}\, dy$

$= 2\int_0^4 (y+2)^{1/2}\, dy$

$= 2\left[\frac{(y+2)^{3/2}}{\left(\frac{3}{2}\right)}\right]_0^4$

$= \frac{4}{3}\left[(y+2)^{3/2}\right]_0^4$

$= \frac{4}{3}\left[(6)^{3/2} - 2^{3/2}\right]$

$= \frac{4}{3}\left[6.\sqrt{6} - 2.\sqrt{2}\right] = \frac{8}{3}\left(3\sqrt{6} - \sqrt{2}\right) u^2$

4	Correct response
3	One error in correct method or Found area between curves but find not remove the excess area between $-\sqrt{2} \le x \le \sqrt{2}$
2	Two errors with correct procedure or error or incomplete solution with the method of area between two curves
1	Finding the 2 lots of the area under the curve $y = x^2 - 2$ between $\sqrt{2}$and$\sqrt{6}$

Markers comments:

Question 4 (9 marks) Start a new writing booklet **Marks**

(a) The first term of a geometric series is e^x and the fifth term is $81e^{9x}$.

(i) Show that the common ratio can be expressed by $r = 3e^{2x}$. **2**

(ii) Find an expression for the 8th term of the series. **1**

(iii) For what values of x does the series have a limiting sum? **3**

(iv) Find the exact value of x if the limiting sum is 3. **3**

i) $T_n = ar^{n-1}$ $T_5 = e^x r^4 = 81e^{9x}$

$a = e^x$ $r^4 = 81e^{8x}$

$r = \sqrt[4]{81e^{8x}}$

$r = 3e^{2x}$

ii) $T_8 = ar^7 = e^x (3e^{2x})^7$

$= e^x . 2187 e^{14x}$

$= 2187e^{15x}$

iii) Limiting sum exists if $|r| < 1$

$|3e^{2x}| < 1$ but $3e^{2x} > 0$ for all x anyway

$-1 < 3e^{2x} < 1$

$0 < 3e^{2x} < 1$ $\Leftarrow$ so

$0 < e^{2x} < \frac{1}{3}$

$2x < \ln(\frac{1}{3})$

$x < \frac{1}{2} \ln(\frac{1}{3})$

Markers comments:

Markers comments:

iv) $S_\infty = 3 = \frac{a}{1-r}$

$$3 = \frac{e^x}{1-3e^{2x}}$$

$$3(1-3e^{2x}) = e^x$$

$$3 - 9e^{2x} = e^x$$

$$0 = 9e^{2x} + e^x - 3$$

let $u = e^x$ $\quad 0 = 9u^2 + u - 3$

$$u = \frac{-1 \pm \sqrt{1-4(9)(-3)}}{18}$$

$$u = \frac{-1 \pm \sqrt{1+108}}{18}$$

$$u = \frac{-1 \pm \sqrt{109}}{18}$$

not u can't be negative

$$e^x = \frac{1+\sqrt{109}}{18}$$

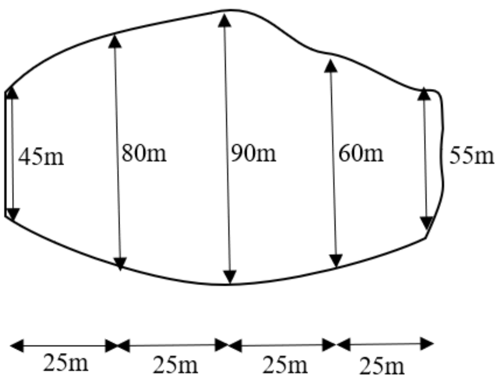
$$x = \ln\left(\frac{-1+\sqrt{109}}{18}\right)$$

$$x = -0.65$$

End of Question 4

Question 5 (7 marks) Start a new writing booklet **Marks**

(a) A factory located next to a lake has recently been shut down due to concerns about chemical run off. A bird's eye view of the lake is given in the diagram below.



Local volunteers have begun a clean-up effort to remove these harmful chemicals from the lake. The concentration of chemicals is given by the equation:

$$C = 1.5 - 0.4e^{kt}$$

Let C be the concentration of chemicals in the lake in kg/m^3, where k is a constant and t is the number of years after the clean-up effort has started.

(i) The initial weight of chemical run off in the lake is 38500kg, and the average depth of the lake is 5m. Use the trapezoidal rule to show that the initial concentration of chemical run off can be estimated as 1.1 kg/m^3. **3**

(ii) Find the value of k to three significant figures if it takes 2.45 years to remove all chemical run off from the lake **2**

(iii) Find the rate of change in the concentration of chemical run off in the lake at $t = 1.8$ years. Round your answer to two decimal places. **2**

i) $A = \frac{25}{2}(45+55+2(80+90+60))$

$= 7000\,m^2$

So $V = 7000 \times 5$

$= 35000\,m^3$

and concentration is $\frac{38500\text{ kg}}{35000\,m^3} = 1.1\ \frac{kg}{m^3}$

ii) $C = 1.5 - 0.4e^{kt}$

$0 = 1.5 - 0.4e^{k(2.45)}$

$\frac{1.5}{0.4} = e^{2.45k}$

$2.45k = \ln(1.5/0.4)$

$k = \frac{\ln(1.5/0.4)}{2.45}$

$k = 0.539$ (3 sig. fig.)

iii) $C = 1.5 - 0.4e^{0.539t}$

$\frac{dc}{dt} = -0.4 \times 0.539e^{0.539t}$

when $t = 1.8$

$\frac{dc}{dt} = -0.4 \times 0.539e^{0.539 \times 1.8}$

$= -0.5699\ldots$

$\doteqdot -0.57$ kg/m³ per year (2 d.p.)

i) 1 mark for applying trapezoidal rule correctly

1 mark for finding volume of lake

1 mark for finding concentration

ii) 1 mark for working towards correct answer

1 mark for correct answer

iii) 1 mark for working towards correct answer

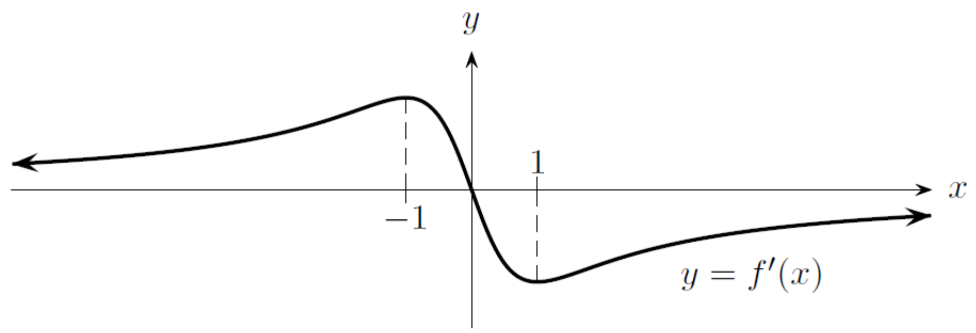
1 mark for correct answer

Markers comments:

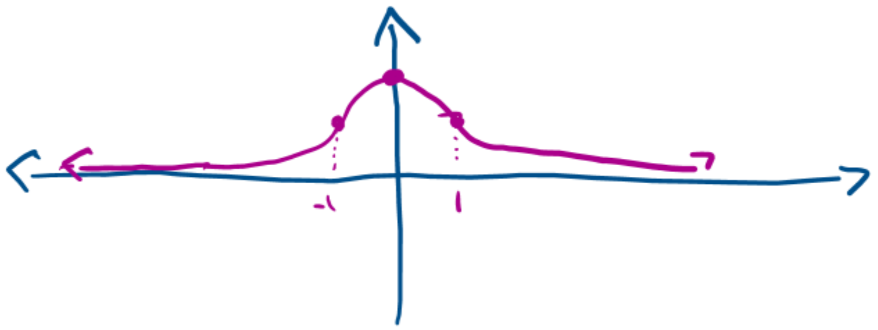
End of Question 5

Question 6 (10 marks) Start a new writing booklet **Marks**

(a) The graph of $y = f'(x)$ is shown in the diagram below. $y = f'(x)$ passes through the origin. As $x \to \pm\infty$, $f'(x) \to 0$ and $f(x) \to 0$. **3**



Sketch the graph of $y = f(x)$, given $f(x) > 0$ for all values of x.



Max Turning point at $x=0$

Inflection point at $x = \pm 1$

Asymptote along $y=0$

1 mark for max TP when $x = 0$

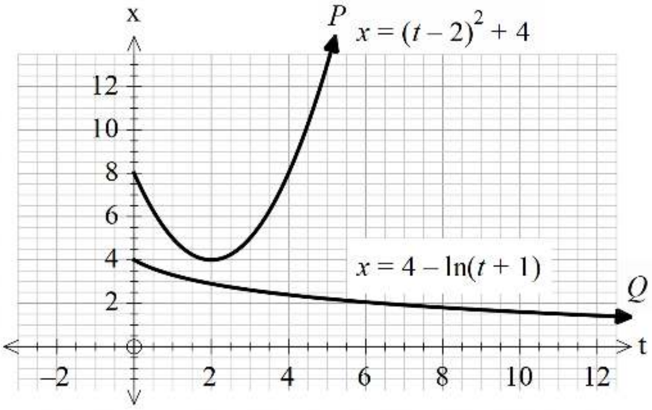
1 mark for inflection points at $x = -1, x = 1$

1 mark for asymptote along $y = 0$

Markers comments:

(b) Two particles P and Q start moving along the x axis at time $t = 0$. Particle P is initially at $x = 8$ and its velocity v in m/s at time t in seconds is given by $v = 2t - 4$.

The position of particle Q is given by $x = 4 - \ln(t+1)$. The diagram shows the graph of $x = 4 - \ln(t+1)$.



(i) Show that the position of particle P is given by $x = (t-2)^2 + 4$. **2**

(ii) Explain why particles P and Q will never collide. **1**

(iii) Show that the distance between the particles can be given by $PQ = (t-2)^2 + \ln(t+1)$. **1**

(iv) Find the exact time that the distance between the particles PQ is a minimum. **3**

i) $v = 2t - 4$
$x = t^2 - 4t + C$
when $t = 0$, $x = 8$
$8 = 0 - 0 + C$
$C = 8$
so $x = t^2 - 4t + 8$
$= (t-2)^2 + 4$

ii) The displacement - time graph for P and never intersects.

i) 1 mark integrating correctly and finding C

1 mark for completing the square to give $x = (t-2)^2 + 4$

ii) 1 mark for correct answer

iii) 1 mark for correct answer

iv) 1 mark for differentiating correctly and expressing as a quadratic

1 mark for finding t

1 mark for checking solution with PQ"

iii) $PQ = \left(\left((t-2)^2 + 4\right) - \left(4 - \ln(t+1)\right)\right)$

$= (t-2)^2 + 4 - 4 + \ln(t+1)$

$= (t-2)^2 + \ln(t+1)$

iv) $PQ' = 2t - 4 + \frac{1}{t+1} = 0$

$2t - 4 = \frac{-1}{t+1}$

$(2t-4)(t+1) = -1$

$2t^2 + 2t - 4t - 4 + 1 = 0$

$2t^2 - 2t - 3 = 0$

$t = \frac{2 \pm \sqrt{4 - 4(2)(-3)}}{4}$

$t = \frac{2 \pm \sqrt{28}}{4}$

$t = \frac{2 \pm 2\sqrt{7}}{4}$

$t = \frac{1 \pm \sqrt{7}}{2}$, $t \geq 0$

so $t = \frac{1+\sqrt{7}}{2}$ s

$PQ' = 2t - 4 + (1+t)^{-1}$

$PQ'' = 2 - (1+t)^{-2}$

when $t = \frac{1+\sqrt{7}}{2}$

$PQ'' = 2 - \frac{1}{\left(1 + \frac{1+\sqrt{7}}{2}\right)^2}$

$PQ'' \approx$ ~~[illegible]~~ $1.875 > 0$

so the distance is a minimum when $t = \frac{1+\sqrt{7}}{2}$ s

Markers comments:

End of Examination